Sums and Products

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For the sake of sake of concreteness, you can define the sum of an array as follows.

```
sum([]) = 0
sum([x1,x2...xn]) = x1 + sum [x2...xn]
```

It's may be useful to consider and prove some formal properties of sums. For example, if S is a finite indexing set,

- $\Sigma_{i \in S} a_i + b_i = (\Sigma_{i \in S} a_i) + (\Sigma_{i \in S} b_i)$
- $\Sigma_{i \in S} k a_i = k \Sigma_{i \in S} a_i$
- $\Sigma_{i \in S} a_i = \Sigma_{j \in T} a_{f(j)}$ if $f : T \to S$ is a bijection
- $\Sigma_{i=0}^{N} i = \frac{1}{2} N(N+1)$
- $\Sigma_{i=0}^{N} i^2 = \frac{1}{6} N(2N+1)(N+1)$
- $\Sigma_{i=0}^{N} i^3 = \frac{1}{4} N^2 (N+1)^2$
  - This one has several corollaries.
- $\Sigma_{i=0}^{N} r^i = \frac{1-r^{N+1}}{1-r}$
- $\Sigma_{i=0}^{\infty} r^i = \frac{1}{1-r}$
  - Using Taylor Series and using finite geometric series.

Next try some products and open-ended sums.

- $\Pi_{k=1}^{N} a^k$
- $\Pi_{k=10}^{30} a^k / b$
- $\lim_{n \to \infty} \frac{1}{n} \Sigma_{k=0}^{n-1} e^{k/n}$
- $\Sigma_{i=0}^{10} \Sigma_{j=0}^{i} (i + 2j + 1)$
- $\Sigma_{i=0}^{100} \Sigma_{j=0}^{100} (i + j)^2$

Now for some difficult sums that can be done using tricks etc.

- "Telescoping"
  - $\Sigma_{n=2}^{\infty} \frac{1}{n(n+1)}$
  - $\Sigma_{n=2}^{\infty} \frac{2n+1}{n^4+2n^3+n^2}$
- $\Sigma_{n=1}^{\infty} \frac{n}{2^n}$