Proofs Techniques

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1 Direct Proofs

Direct proofs are the most simple and intuitive of proofs. Simply assume p, then demonstrate that q must hold, and you have show $p \to q$. Always contemplate direct proofs before you think about other techniques.

Examples:

- If $\binom{n}{r} = \frac{n!}{r!(n-r)!}$, then $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$
- If $0 < a < b$, then $a < \sqrt{ab} < \frac{a+b}{2} < b$. This one may be slightly less direct than usual, because you may want to start and the results and show that they are equivalent to true (whenever the hypothesis is satisfied).
- The base ten number $a_n a_{n-1} ... a_1 a_0$ is divisible by 3 if and only if $a_n + a_{n-1} + ... + a_1 + a_0$ is divisible by 3.

2 Proof by Contrapositive

In this proof strategy, you prove $\neg q \to \neg p$, which is equivalent to $p \to q$. This is closely linked to the next section, proof by contradiction.

- If k^2 is even, k is even.
- If $x + y$ is even, either x and y are both odd or x and y are both even.
- If $n \equiv 2$ or $n \equiv 3 \bmod 4$, then n is not a perfect square.

3 Proof by Contradiction

Here, you simply assume $\neg p$ and derive a contradiction, thus showing p. More precisely, you must show a derive a contradiction in order to show that $\neg p \to F$, which is equivalent to $T \to p \equiv p$.

- There are an infinite number of primes (hint: assume otherwise and consider the number $p_1 \cdot ... \cdot p_n + 1$).
- $\sqrt[n]{a}$ is irrational unless a is an n^{th}power.

4 Proof by Cases

If we want to show that $p \to q$, and can break down our hypothesis into a number of cases such that $p \equiv p_1 \lor p_2 \lor ... \lor p_n$, then we can prove $(p_1 \to q) \land (p_2 \to q) \land ... \land (p_n \to q)$. This equivalence itself can be proven (by induction if you are feeling fancy).

- $|x+y| \leq |x| + |y|$. Hard way: do a bunch of casework on the sign of x and y and their sum. Easy way: try squaring both sides and use the fact that the square of a number is always nonnegative.
- $\lfloor \frac{x+1}{2} \rfloor - 1 = \lfloor \frac{x-1}{2} \rfloor$. Hint: odd and even numbers behave in different ways when divided by 2.