Midterm 1 Review

Clay Thomas

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1 Relations

Classify whether the following relations are symmetric, transitive, reflexive, or antisymmetric. If they are equivalence relations, partial orders, or total orders, state so. If they are equivalence relations, describe the equivalence classes.[1]

1. $fRg \iff (f-g)(x) > 0$ for all $x \in \mathbb{R}$
2. $(a,b)R(c,d) \iff ad = bd$
3. $(a,b)R(c,d) \iff ac = bd$
4. Given a relation $\trianglelefteq$ that is known to be reflexive and transitive (but not necessarily antisymmetric or anything else), let $aRb \iff a \trianglelefteq b \wedge b \trianglelefteq a$.
5. $fRg \iff f$ is $O(g)$

2 Proofs

Prove (using whatever technique you deem best, but be rigorous (no Venn Diagrams)) the following:

1. $\Sigma_{k=1}^{N} k^p$ is $\Theta(N^{p+1})$
2. $\frac{x^4+2x^2+5}{x^4+1}$ is $O(1)$
3. $(x+y)^n = \Sigma_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$
4. $|x+y| \leq |x| + |y|$
5. If k^2 is even, k is even.
6. $A - B \subseteq A$
7. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

[1] In each of these problems, we use a shorthand to describe R. $aRb \iff P(a,b)$ is meant to mean that $(a,b) \in R$ if and only if $P(a,b)$. In other words, if aRb, then a is related to b in the way logically specified by the right hand side of the double arrow.

8. $(p \wedge q) \vee (p \wedge \neg q) \to p$ is a tautology.

9. $(p \vee q) \wedge (p \to r) \to q \vee r$ is a tautology. Also state the equivalent rule of deduction.

10. If $a|x$ and $a|y$, then $a|(bx + cy)$

11. If n is composite, n has a prime factor $p \leq \sqrt{n}$.

3 Misc

Evaluate the following (with justification)

1. $\Sigma_{i=4}^{n} 2i^2 + 4i + 1$ for $n \geq 4$. You may take the formulas for the sum of the powers as given.

2. Consider $a_n = \frac{1}{2} a_{n-1}$ (with general base case $a_o = c$). Find $\Pi_{k=0}^{N} a_k$.

3. Let $f(x) = \sin(x)$. Find $f(\mathbb{R})$ and $f^{-1}([0, 1)) = f^{-1}(\{x | 0 \leq x < 1\})$

4. Express the following statement using quantifiers: One is the unique multiplicative identity (the number such that ax=x).

5. Are there integers x, y, z such that $6x + 9y + 15z = 107$?