Functions

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Functions are simply rules for taking values and producing new ones. Specifically, a function from a set A to B takes an arbitrary a in A and produces a single b in B.

A few definitions:

- The domain of a function is the set of input values.
- The codomain is the set that the function maps into.
- The image (sometimes range) is the subset of the codomain for which some input actually produces that output.
- A function is injective (one-to-one) when f(a)=f(b) implies a=b.
- A function is surjective (onto) if its image is the entire codomain.
- A function is bijective if it is injective and surjective.
- There exists a bijection between two finite sets if and only if they have the same number of elements. Thus, we say that two infinite sets have the same number of elements if there exists a bijection between them
- Let id_A stand for the identity function on any set A, ie the function such that $id_A x = x$ for every $x \in A$.

First, some easier examples:

- Let $f : \mathbb{N} \to \mathbb{N}$ such that $f(x) = \begin{cases} x+1 & \text{x is even} \\ x-1 & \text{x is odd} \end{cases}$. Then f is a bijection.
- Let $f : [0,1] \to [0,2]$ be such that $f(x) = 2 * x$. Then f is a bijection. There are thus the same number of elements in $[0,1]$ as in $[0,2]$. Email John Green about it at your earliest convenience.
- Let $f(x) = x^2$ and $\mathbb{R}^{\geq 0} = \{x | x \in \mathbb{R} \wedge x \geq 0\}$.
  - If we consider $f : \mathbb{R} \to \mathbb{R}$, then f is neither surjective nor injective.
  - However, if we consider $f : \mathbb{R}^{\geq 0} \to \mathbb{R}$ then f is injective.

- If we consider $f : \mathbb{R} \to \mathbb{R}^{\geq 0}$, f is surjective.
- Finally, if we consider $f : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$, then f is a bijection.

Now for some important results, whose proofs are all left as exercises (or given in PSO):

- If $f : A \to B$ is surjective, there exists a map $g : B \to A$ with $f \circ g = id_B$.
- If $f : A \to B$ is injective, there exists a map $g : B \to A$ with $g \circ f = id_A$
- If $f : A \to B$ is bijective, there exists a **unique** map $g : B \to A$ with $f \circ g = id_B$ and $g \circ f = id_B$.
- There exists a bijection between two finite sets if and only if they have the same number of elements.
- Every function$f : A \to B$ can be expressed as $f = i \circ b \circ s$, where i is injective, b is bijective, and s is surjective.
- There exist bijections between $\mathbb{N}$ and many other sets, including $\mathbb{Z}$ and $\mathbb{Q}$.
- There does not exist a bijection between $\mathbb{N}$ and $\mathbb{R}$. In this sense, there are "more" real numbers than there are rational numbers.
- There never exists a bijection between a set and its power set. In other words, $f : A \to \mathcal{P}(A)$ is always injective.