Maximally Recoverable Codes: the Bounded Case

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Joint work with
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Motivation - Distributed Storage

Row failures | Column failures | Arbitrary failures
Motivation - Distributed Storage

Row failures

Column failures

Arbitrary failures
Motivation - Distributed Storage

Row failures  Column failures  Arbitrary failures
Motivation - Distributed Storage

Row failures
Column failures
Arbitrary failures
Designing a Code

Typical failures are either small OR large but structured (Correlated Erasures)

[CHL07] [Gopalan, Hu, Kopparty, Saraf, Wang, Yekhanin (SODA 17)]

1. Design the topology
   - Known failure patterns
   - Heuristics; hardware
   - Set limits of recoverability

2. Set the coefficients
   - Maximize recoverability
   - Pure math
The Code Topology $T(1, 1, 1)$ [GHK$^+$17]

Design the topology

Row parities

$\text{Design the topology}$

$\text{Row parities}$
The Code Topology $T(1, 1, 1)$ [GHK$^+17$]

Design the topology

<table>
<thead>
<tr>
<th>$u_1$</th>
<th>$v_1$</th>
<th>$w_1$</th>
<th>$u_1+v_1+w_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_2$</td>
<td>$v_2$</td>
<td>$w_2$</td>
<td>$u_2+v_2+w_2$</td>
</tr>
<tr>
<td>$u_1+u_2$</td>
<td>$v_1+v_2$</td>
<td>$w_1+w_2$</td>
<td></td>
</tr>
</tbody>
</table>

Column parities
The Code Topology $T(1, 1, 1)$ [GHK$^+$17]

Maximize Recoverability

One extra, global redundancy

Where $c$ is some linear function of the data symbols
$T(a,b,h)$ [GHK$^+$17]

$T(a, b, h)$:
- $a$ column parities
- $b$ row parities
- $h$ global parities

Facebook: $T(1, 4, 0)$ [SLR$^+$14]

Microsoft: $T(0, 1, 2)$ [HSX$^+$12]

Local Codes: $T(1, 0, h)$ [GHSY12]
The Code Topology $T(1, 1, 1)$ [GHK$^+$17]

View codewords as an $m \times n$ grid of symbols $x_{i,j}$

- Rows are codewords in $C_{\text{Row}}$
- Columns are codewords in $C_{\text{Col}}$
- Satisfy global constraint that

$$0 = \sum_{i \in [m], j \in [n]} \gamma_{i,j} x_{i,j}$$

- WLOG $C_{\text{Row}}$ and $C_{\text{Col}}$ are parity checks
  - entries sum to 0
- A code instantiating $T(1, 1, 1)$ is defined by the constants $\gamma_{i,j}$.
Decoding Examples

<table>
<thead>
<tr>
<th>X</th>
<th>X</th>
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<th>X</th>
<th>X</th>
<th>X</th>
<th>X</th>
</tr>
</thead>
<tbody>
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<td></td>
<td>X</td>
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<td>X</td>
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</tr>
</tbody>
</table>
Decoding Examples

\[ \begin{array}{cccccc}
\times & \times & \times & \times & \times & \times \\
\times & & & & & \\
& \times & & & & \\
\end{array} \Rightarrow \]

\[ \begin{array}{cccccc}
\times & \times & \times & \times & \times & \times \\
& & & & & \\
& & & & & \\
\end{array} \]
Decoding Examples

\[ \begin{array}{cccccc}
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\end{array} \Rightarrow
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\end{array} \Rightarrow
\begin{array}{cccc}
X & X & X & X \\
X & X & X & X \\
\end{array} \]
Decoding Examples
Decoding Examples

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\times & & & & & \\
\end{array}
\]

\[\Rightarrow\]

\[
\begin{array}{cccccc}
\times & & & & & \\
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\times & & & & & \\
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\times & & & & & \\
\end{array}
\]

\[\Rightarrow\]

\[
\begin{array}{cccccc}
\times & & & & & \\
\times & \times & & & & \\
\times & & & & & \\
\times & \times & \times & \times & \times & \times \\
\times & & & & & \\
\end{array}
\]
Decoding Examples

![Decoding Example 1]

![Decoding Example 2]
Maximal Recoverability \cite{GHK+17}

- Recall $C$ instantiates $T(1,1,1)$ by setting the coefficients $\gamma_{i,j}$.

- As code $C$ instantiating $T(1,1,1)$ corrects an erasure pattern $E \subseteq [n] \times [n]$ if the symbols $x_{i,j}$ for $(i,j) \in E$ can be recovered from those in $[n] \times [n] - E$.

- $E \subseteq [n] \times [n]$ is correctable for $T(1,1,1)$ if there is some code instantiating $T(1,1,1)$ which corrects $E$.

- A code for $T(1,1,1)$ is maximally recoverable (MR) if it can correct every correctable erasure pattern.
  - Good news: they exist \cite{CHL07}
  - Bad news: require $\gamma_{i,j} \in \mathbb{F}_2^d$ for $d$ linear in $n$.
    - \cite{Kane,Lovett,Rao(FOCS17)} \cite{GHK+17}
e-Maximal Recoverability

- An e-MR code for a topology corrects all correctable erasure patterns of size \( \leq e \)
  - For \( T(1, 1, 1) \) and constant \( e \), attain field size polynomial in \( n \)
Reducing to labeling problem \([\text{GHK}^+17]\)

Subsets of code symbols $\leftrightarrow$ Sets of vertices in $K_{n,n}$
Symbol $(i, j)$ erased $\leftrightarrow$ Edge $(i, j) \in K_{n,n}$
Parity check weights $\gamma_{i,j}$ $\leftrightarrow$ Edge weight $\gamma(i, j) \in \mathbb{F}_2^d$
(Irreducible) correctable pattern $\leftrightarrow$ Simple cycle
with nonzero weight
Reducing to labeling problem [GHK$^+17$]

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- (Irreducible) correctable pattern $\leftrightarrow$ Simple cycle with nonzero weight

$$
\begin{array}{c}
1 & \rightarrow & 1 \\
2 & \rightarrow & 2 \\
3 & \rightarrow & 3 \\
4 & \rightarrow & 4 \\
5 & \rightarrow & 5 \\
\end{array}
$$
Reducing to labeling problem [GHK+17]

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<th>4</th>
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Reducing to labeling problem \([\text{GHK}^+17]\)

Subsets of code symbols \(\longleftrightarrow\) Sets of vertices in \(K_{n,n}\)
Symbol \((i, j)\) erased \(\longleftrightarrow\) Edge \((i, j) \in K_{n,n}\)
Parity check weights \(\gamma_{i,j}\) \(\longleftrightarrow\) Edge weight \(\gamma(i, j) \in \mathbb{F}_2^d\)
(Irreducible) correctable pattern \(\longleftrightarrow\) Simple cycle with nonzero weight
Reducing to labeling problem

Theorem [GHK$^+$17]: A code with $\gamma(i,j) \in \mathbb{F}_2^d$ corrects an (irreducible) error $E \subseteq [n] \times [n]$ iff $E \subseteq K_{n,n}$ is a simply cycle with

$$0 \neq \gamma(E) := \sum_{(i,j) \in E} \gamma(i,j)$$

- Correcting unbounded-length cycles requires $d = \Theta(n)$ [KLR17]
- Our observation: $|\mathbb{F}_2^d|$ polynomial in $n$ if you only correct cycles of bounded length
Our Problem

Given $n, e$.
Find a labeling $\gamma : [n] \times [n] \to \mathbb{F}_2^d$ such that for all simple cycles $E$ in $K_{n,n}$ of length at most $e$, we have $\gamma(E) \neq 0$.
Goal: minimize $d$.

- Handle small (constant) number of arbitrary erasures
- Implies existence of $e$-MR code
Our Results

(Asymptotic) bounds on $|\mathbb{F}_2^d|$ for $e$-MR codes on $n \times n$ codewords

<table>
<thead>
<tr>
<th>$e$</th>
<th>u.b.</th>
<th>l.b.</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>$n$</td>
<td>$n$</td>
</tr>
<tr>
<td>6</td>
<td>$n^2$</td>
<td>$n^2$</td>
</tr>
<tr>
<td>8</td>
<td>$n^3$</td>
<td>$n^2$</td>
</tr>
<tr>
<td>10</td>
<td>$n^4$</td>
<td>$n^3$</td>
</tr>
<tr>
<td>12</td>
<td>$n^5$</td>
<td>$n^3$</td>
</tr>
</tbody>
</table>

Previous results:

- $\leq n^e$ (implied in [GHJY14])
- $\geq \Omega((n/e)^{\log(e/2)})$ for $e \leq \sqrt{n}$ (implied in [GHK+17])
Edge labeling for $e = 4$

Take $\{P_i\}$ and $\{Q_j\}$ each distinct and let

$$\gamma(i, j) = P_i Q_j \in \mathbb{F}_{2d} \cong \mathbb{F}_2^d$$

Only need $n$ distinct $\{P_i\}, \{Q_j\} \subseteq \mathbb{F}_2^d$ so $|\mathbb{F}_2^d| = O(n)$.

$$\gamma(E) = \gamma(i_1, j_1) + \gamma(i_1, j_2)$$
$$+ \gamma(i_2, j_1) + \gamma(i_2, j_2)$$
$$= P_{i_1} Q_{j_1} + P_{i_1} Q_{j_2}$$
$$+ P_{i_2} Q_{j_1} + P_{i_2} Q_{j_2}$$
$$= (P_{i_1} + P_{i_2})(Q_{j_1} + Q_{j_2})$$
$$\neq 0$$
Edge labeling for $e = 8$

Take \( \{P_i\} \) and \( \{Q_j\} \) each distinct and let

\[
\gamma(i,j) = (P_iQ_j, P_i^2Q_j, P_i^4Q_j) \in \mathbb{F}_{2^{\log n}}^3 \cong \mathbb{F}_2^{3\log n}
\]

\[
\gamma(E) = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_1^2 & a_2^2 & a_3^2 \\ a_1^4 & a_2^4 & a_3^4 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}
\]

Where $a_m = P_{i_1} + P_{i_m+1}$ and $b_m = Q_{j_m} + Q_{j_{m+1}}$. 
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$$\gamma(E) = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_2 & a_2 & a_2 \\ a_3 & a_4 & a_4 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Where $a_m = P_{i_1} + P_{i_{m+1}}$ and $b_m = Q_{j_m} + Q_{j_{m+1}}$. 
Lower Bound for $e = 4$

Theorem: at least $n$ different labels in $\mathbb{F}_2^d$ are required.
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Theorem: at least $n$ different labels in $\mathbb{F}_2^d$ are required.

Proof: If $1^2_3$ and $1^2_3$ have the same weight, then $1^2_3$ is a simple cycle with zero weight.
Lower Bound for $e \leq 12$

Create a graph.

- Vertices: some collection of paths in $K_{n,n}$
- Edges: $p_1 - p_2$ if $p_1$ and $p_2$ together form a simple cycle of length $\leq e$
  - Connected paths must have different $\gamma$ weight
  - $|\mathbb{F}_2^d|$ is at least the chromatic number
  - Observation: a valid labeling induces a valid coloring

For $e = 4$ we build a clique of size $n$

```
1 -- 2 -- 3
|   |   |
|   |   |
4 -- 5
```
Lower Bound for $e \leq 12$

Create a graph.

- Vertices: some collection of paths in $K_{n,n}$
- Edges: $p_1 - p_2$ if $p_1$ and $p_2$ together form a simple cycle of length $\leq e$
  - Connected paths must have different $\gamma$ weight
  - $|\mathcal{F}_2|$ is at least the chromatic number
  - Observation: a valid labeling induces a valid coloring

For $e = 6$ we build a clique of size $(n - 1)^2$
Lower Bound for $e \leq 12$

Create a graph.

- Vertices: some collection of paths in $K_{n,n}$
- Edges: $p_1 - p_2$ if $p_1$ and $p_2$ together form a simple cycle of length $\leq e$
  - Connected paths must have different $\gamma$ weight
  - $|\mathbb{F}_2^d|$ is at least the chromatic number
- Observation: a valid labeling induces a valid coloring

For $e = 10$ we get chromatic number $\Omega(n^3)$
Conclusions and Open Questions

We showed

- \(|\mathbb{F}_2^d| \leq n^{e/2-1}\) for \(e \leq 12\)
  - Tight for \(e = 4, 6\)
- \(|\mathbb{F}_2^d| \geq n^3\) for \(e = 10\)

Open questions

- Tight bounds for all \(e\)
- Construct \(e\)-MR codes for other topologies