The short side advantage in random matching markets

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Preliminaries and Results

Goal: investigate the effect of *competition* on the average case behaviour of stable matching markets

- n men and $n+k$ women for $k \geq 0$
  - ▷ Uniformly random, full length preference lists
- [AKL] found a startling difference between $k = 0$ and $k \geq 1$
  - ▷ We give a new, simpler proof
- Focus on *women's average rank for their husbands*
  - ▷ Smaller is better

Women's average rank for their husbands

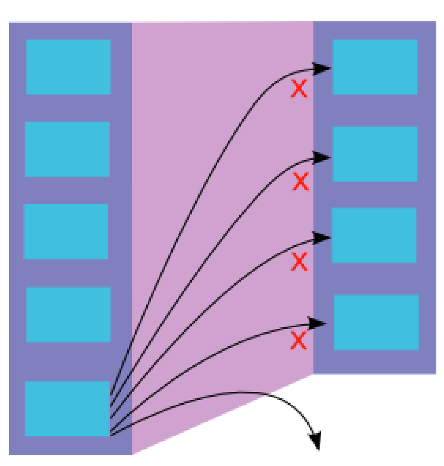
	Men-proposing	Women-proposing
n men, n women [Pit]	$\Omega(n/\log n)$	$O(\log n)$
n men, $n+1$ women [AKL]	$\Omega(n/\log n)$	$\Omega(n/\log n)$

Balanced Market

- **Observation 1:** In women-proposing deferred acceptance ($WPDA$), the sum of women's rank for their husbands equals the total number of proposals.
- **Observation 2:** In balanced market, $WPDA$ terminates as soon as n distinct men are proposed to.
- **Observation 3:** Therefore, the number of proposals made in $WPDA$ is essentially a **coupon collector** random variable.
  - ▷ If numbers from $[n]$ are repeatedly drawn u.a.r., the number of draws needed for every element of $[n]$ to occur gives the coupon collector random variable
  - ▷ In a random market, each proposal women make is essentially uniform over the n men.
  - ▷ The only difference between $WPDA$ and a coupon collector is that women never propose to the same man twice. (Formally, the coupon collector statistically dominates the number of proposals made.)
  - ▷ The expected number of draws the coupon collector needs is $O(n \log n)$.
- **Conclusions:**
  - ▷ In $WPDA$, $O(n \log n)$ proposals are made on average, so women's average rank for their husband is $O(\log n)$.
  - ▷ In $WPDA$, men receive $O(\log n)$ proposals on average, so their rank for their wife is the minimum rank over these proposals, which is $\Omega(n/\log n)$.

Sharp Transition From $n \times n$ to $n+1 \times n$

- When $n+1$ women propose to n men, the random process terminates when some *specific* woman has proposed to every man.
- No woman wants to go unmatched, so they keep proposing to men and pushing each other down their preference lists.



- Shift perspective to *man-proposing deferred acceptance* ($MPDA$).
- Investigate the woman-optimal outcome by finding the best strategic manipulation a woman can achieve.

Lemma ([IM])

A woman w^* has a stable partner of rank better than i if and only if w^* remains matched in man-proposing deferred acceptance when she truncates her list after rank i.

Unbalanced Market

- By the lemma, woman w^*'s rank for her partner in woman-optimal outcome is the best (i.e. minimum) rank i at which she can truncate her list while still being matched in $MPDA$.
- Consider running $MPDA$ with n men and $n+1$ women. Similar to the balanced market, $MPDA$ terminates as soon as n distinct women have been proposed to.
- Now imagine w^* **rejects all proposals she receives**. Run $MPDA$ until all women other than w^* receive a match.
  - ▷ The number of proposals again follows a **coupon-collector** random variable, and we expect $O(n \log n)$ total proposals.
  - ▷ So w^* should get $O(\log n)$ total proposals.
- Thus, the expected best (minimum) rank of a proposal she receives is $\Omega(n/\log n)$. This is also the minimum rank where she can truncate her list and still receive a match, so in expectation she has no stable partners better than this rank.

Larger Imbalance

- Fix constant $\lambda > 0$ and consider n men and $(1+\lambda)n$ women.
- Women's average rank for their husbands is $\Omega(n)$.
  - ▷ More specifically, $\Omega(n/\log(1+1/\lambda))$.

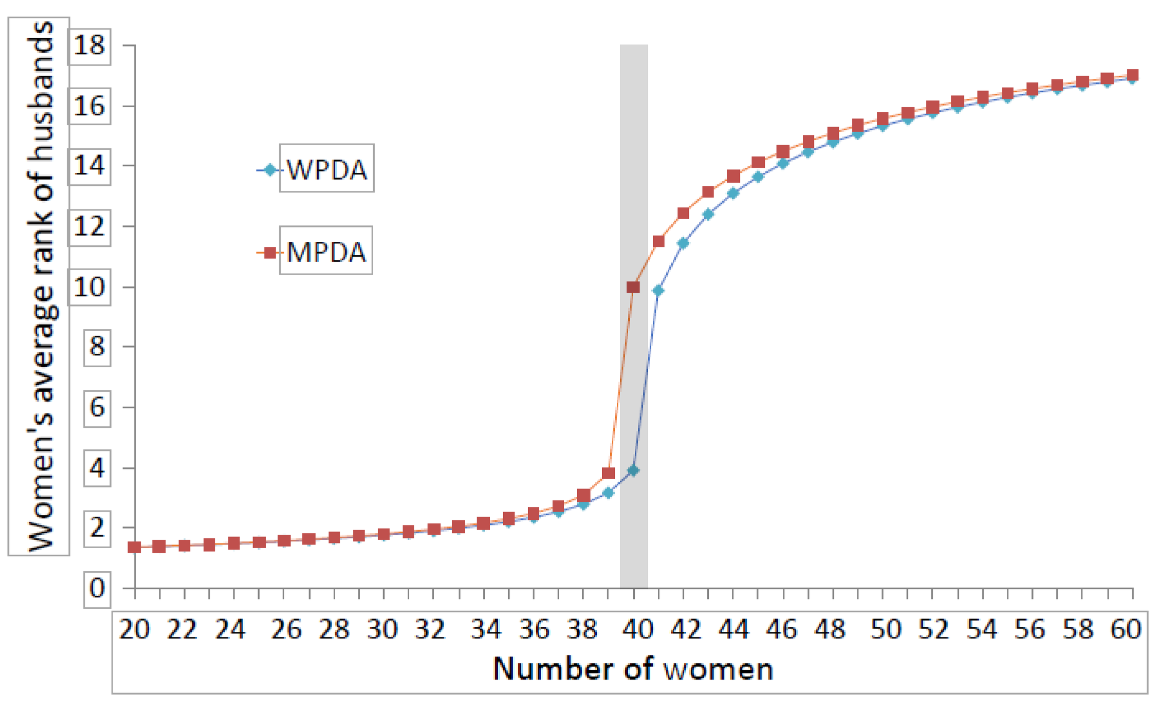


Figure: [AKL] Women's average rank of husbands in WPDA and MPDA.

Citations

AKL Itai Ashlagi, Yash Kanoria, and Jacob D. Leshno. Unbalanced random matching markets: The stark effect of competition. Journal of Political Economy, 125(1):69 – 98, 2017.

IM Nicole Immorlica and Mohammad Mahdian. Marriage, honesty, and stability. SODA 2005, Vancouver, British Columbia, Canada, January 23-25, 2005, pages 53–62, 2005.

Pit Boris Pittel. The average number of stable matchings. SIAM J. Discret. Math., 2(4):530–549, November 1989.7.