# The Short-Side Advantage in Random Matching Markets

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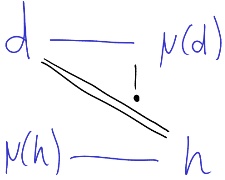
# Overview

- Stable matching market
  - “Doctors” being matched to “hospitals”
  - Each agent has *preferences* $\succ_d$ over the other side
  - Stability of $\mu$: No *unmatched* $d, h$ with $h \succ_d \mu(d)$, $d \succ_h \mu(h)$




- [Ashlagi, Kanoria, Leshno 17]: *imbalance* in the number of agents on each side profoundly effects (average behaviour of) these matchings
  - Even with $n$ doctors and $n + 1$ hospitals
- Our paper: a simple proof of (some of) their results

# Introduction

# Background

- Stable matching markets
  - Stability of $\mu$: No *unmatched* $d, h$ with $h \succ_d \mu(d)$, $d \succ_h \mu(h)$
- Critical in real world two-sided markets
  - Stability prevents “market unraveling” [Roth 2002]
- A vast classic literature investigates structure
  - [Gale and Shapley 1962], [Knuth 77], [Gusfield and Irving 89]
- Always exists a stable matching. In fact, there can be *many*
- How do we pick one?

# Background

- In practice: *doctor-optimal* stable matching used
  - (It turns out this is unique)
- Computed via doctor-proposing **Deferred Acceptance (DA):** (Until everyone matched): Doctors “propose” in order of their preference list, hospitals “tentatively accept” their highest-preference proposal they receive
- Advantages:
  - Simple and fast algorithm
  - Good incentive properties
- Still, choice of doctor-proposing feels arbitrary…

# What matters for the matching?

- How different are the doctor and hospital optimal matchings?
- What determines who gets matched where?

# What matters for the matching?

- [Wilson 72, Pittel 88 & 89]: what matters is *who is proposing*
  - Consider $n$ doctors ranking each of $n$ hospitals
  - Consider (uniformly) random preference lists
  - Proposers get their $\log n$th choice, receivers get $n / \log n$
  - Set of stable matchings is large: Agents have $\log n$ stable partners on average
- [Immorlica-Mahdian 05 & 15]: what matters is *the length of preference lists*
  - Motivated by fact that markets are too big to rank everyone
  - If each agent ranks $k = O(1)$ others (uniformly), then agents have unique stable partners w.h.p.
  - Doesn't matter who proposes!
- **[Ashlagi-Kanoria-Leshno 2017]: what matters is the *balance of the market***

# [AKL]

- [Ashlagi-Kanoria-Leshno 2017]:
  - Say $n$ doctors and $\mathbf{n+1}$ **hospitals**
  - All doctors rank all hospitals (and vice-versa)
  - **Theorem:** Agents have unique stable partners w.h.p.
  - **Theorem:** Doctors get $O(\log n)$th choice, hospitals get $O(n/\log n)$th, *regardless of who proposes*

| (Doctor's $\mathbb{E}$ [rank]) | Doctor-optimal | Hospital-optimal |
|---|---|---|
| $n \times n$ | $O(\log n)$ | $O(n/\log n)$ |
| $n \times (n+1)$ | $O(\log n)$ | $O(\log n)$ |

- Agents on the *short side* at a large advantage
- Our contribution: simpler proofs!

# Intuition

# Deferred Acceptance

- Proposing-side “proposes” in order of their preferences
- Receiving-side “keeps the best proposal they’ve seen so far”
  - “Rejected” agents keep proposing
- Repeat (until all proposers matched or exhaust pref list)
  - **Only way a proposer can go unmatched is if they are rejected by their entire list**

$h_1$ - $d_1$ $d_2$

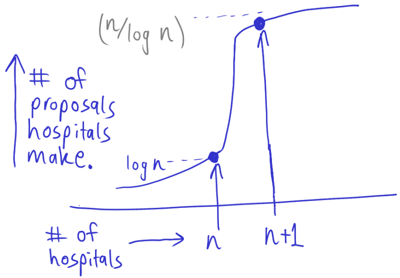
$h_2$ -

$h_3$ - $d_3$ $d_4$

$h_4$ -

# Intuition: a sharp transition

- Consider *hospital* proposing DA
  - Imagine each proposal made at random "online"
- If $n$ hospitals propose to $n$ doctors, *(balanced)*
  $\implies$ terminate when every doctor receives a proposal
- If $n+1$ hospitals propose to $n$ doctors, *(unbalanced)*
  $\implies$ terminate when *some specific hospital proposes to* ***every*** *doctor*
  - No hospital wants to go unmatched, creating "congestion"



# Proof

# Balanced Case

- Analysis with $n$ doctors proposing to $n$ hospitals:
  - Imagine each proposal made at random "online"
  - DA terminates when all $n$ hospitals receive a proposal
  - When $i$ hospital have receive a proposal, the *next* proposal goes to a *new* hospital with probability $(n-i)/n$
  - (Coupon collector)
  - In expectation, this take total proposals:

$$\frac{n}{n} + \frac{n}{n-1} + \frac{n}{n-2} + \ldots \frac{n}{1} = n \cdot H_n \approx n \log n$$

  - Thus, $\log n$ proposals (i.e. average rank) per doctor

$i$ proposed-to hospitals $\Longrightarrow$ $\frac{n-i}{n}$ chance of proposing to a new hospital

# Lemma: [Immorlica, Mahdian 05]

- (Rural Hospital / Lone Wolf) Theorem: the set of matched agents is the same in ever stable matching
- **Proposition:** A hospital *h* has a stable parter of rank better than *i* $\iff$ In (doctor proposing) DA, *h* receives a match *even if h truncates their list after rank i*
  - ($\Longleftarrow$) (Fairly easy to check) if *h* matched and $\mu$ stable for truncated preferences, then $\mu$ stable for original prefs
  - ($\Longrightarrow$) Similar, using Rural Hospital Theorem

$h: \; d_1, \dots \quad d_i, \boxed{d_{i+1}, \dots}$

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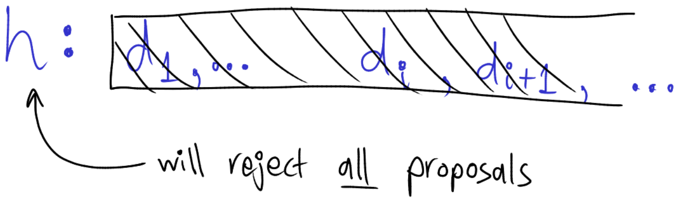
- **Lemma:** Consider doctor-proposing DA, where *h truncates their entire list*. Then *h*'s rank in *hospital optimal* match is the rank of the best proposal they receive.



# Main Proof

- **Lemma:** Consider doctor-proposing DA, where *h truncates their entire list*. Then *h*'s rank in *hospital optimal* match is the rank of the best proposal they receive.
- Consider $n$ (proposing side) doctors and $n+1$ hospital
- If *h*'s list is empty, DA behaves *essentially like the balanced case*
  - Terminates when $n$ distinct non-$h$ hospitals proposed to
  - $n \log n$ proposals total, i.e. $\log n$ per hospital
- In expectation, the best of these $\log n$ random proposals is *h*'s rank $(n/\log n)$th choice
- $\implies$ **Theorem:** hospital get no better than $n/\log n$, even in hospital optimal outcome

# Extensions

- New question: *number of distinct stable partners?*
- Consider $n$ (proposing side) doctors and $n+1$ hospital
- Consider DA, where *h truncates their entire list*
- $\implies \mathbb{P}[h \text{ has multiple stable partners}] = \mathbb{P}[h\text{'s favorite prop came after } n-1 \text{ hospital prop'ed to}]$
  - In expectation, $\Omega(\log(n))$ proposals *before* $n-1$ hospitals proposed to, and $O(1)$ proposals after
  - $\implies \mathbb{P}[\cdot] = O(1/\log n)$
- **Theorem:** An agent has a unique stable partner w.h.p.
- (From here you can also bound doctor's ranks)

# Another intuition

- With $n$ doctors and $n+1$ hospitals, a hospital is *essentially unneeded* to form the matching
  - Settles for a partner "only $\log n$ better than random"
- [AKL] study "gap between doctor and hospital optimal"
  - Very powerful but complicated
- Our proof directly studies the hospital optimal

# Conclusion

- Lots of factors effect the market!
  - Our focus: balance.
  - Mentioned short lists
- [Kanoria, Min, Qian 20]: Short lists *and* imbalance
- [Gimbert, Mathieu, Mauras 20],
  [Ashlagi, Braverman, Saberi, Thomas, Zhao 21]:
  models of a-priori *quality* of agents
- [Beyhaghi, Tardos 21]: interview matchings
- Still gaps in our understanding!
  - Motivating question: why do people apply to “a few reach schools, several reasonable choices, and a safety school”?