A FUNKY FUNCTION ON THE FRACTIONS: POTW 13 (SPRING 2015)

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Theorems

We are given that $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ satisfies $f(u \cdot f(v)) = f(u)/v$ for every $u, v \in \mathbb{Q}^+$. We proceed with a series of lemmas and results. One quite nice thing about this problem is that we are considering only positive rationals and thus all numbers below are nonzero and have multiplicative inverses.

Lemma 1. *$f(f(x)) = f(1)/x$ for all $x \in \mathbb{Q}^+$.*

Proof. We have $f(f(x)) = f(1 \cdot f(x)) = f(1)/x$. □

Lemma 2. *$f(z) = 1$ if and only if $z = 1$.*

Proof. Using Lemma 1, we see

$$f(1) = f(1)/1 = f(f(1)) = f(u \cdot f(1)/u) = f\big(u \cdot f(f(u)\big) = f(u)/f(u) = 1$$

Thus $z = 1$ implies $f(z) = 1$.

Conversely, assume $f(z) = 1$. Then we have

$$1/z = f(z)/z = f(z \cdot f(z)) = f(z \cdot 1) = 1$$

Thus we get $z = 1$. □

Corollary 3. *$f(f(x)) = 1/x$ for all $x \in \mathbb{Q}^+$.*

Proof. Lemmas 1 and 2. □

Lemma 4. *$f(x/y) = f(x)/f(y)$ for all $x, y \in \mathbb{Q}^+$*

Proof. By Corollary 3,

$$f(x \cdot 1/y) = f\big(x \cdot f(f(y))\big) = f(x)/f(y)$$

□

Theorem 5. *f is injective.*

Proof. Assume that $f(x) = f(y)$. By Lemma 4 we have

$$1 = f(x)/f(y) = f(x/y)$$

By Lemma 2 we now know $x/y = 1$, so $x = y$ as desired. □

Theorem 6. *f is surjective.*

Proof. Given a $z \in \mathbb{Q}^+$ we need to find an x such that $f(x) = z$. Well,

$$f(1/f(z)) = f(1)/f(f(z)) = 1/(1/z) = z$$

Thus every element of the codomain is mapped to by some element of the domain. □

Corollary 7. *f is bijective and satisfies $f^{-1}(z) = 1/f(z)$ for all $z \in \mathbb{Q}^+$.*

Proof. Theorems 5 and 6 tell us f is invertible, and the proof of Theorem 6 shows that $1/f(z)$ gets mapped to z under f. Thus $f^{-1}(z) = 1/f(z)$. □

Theorem 8. *$f(xy) = f(x)f(y)$ for all $x, y \in \mathbb{Q}^+$.*

Proof. Making use of Corollary 7,

$$f(xy) = f\Big(x \cdot f(f^{-1}(y))\Big) = f(x)/f^{-1}(y) = f(x)/\big(1/f(y)\big) = f(x)f(y)$$

□

Construction

We now give a somewhat convoluted construction of an example function F that satisfies the requirement $F(u \cdot F(v)) = F(u)/v$ for all $u, v \in \mathbb{Q}^+$. First, we provide a somewhat simpler condition for f which is equivalent.

Theorem 9. *If $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ satisfies $f(f(x)) = 1/x$ and $f(xy) = f(x)f(y)$ for all $x, y \in \mathbb{Q}^+$, we have $f(u \cdot f(v)) = f(u)/v$ for all $u, v \in \mathbb{Q}^+$.*

Proof. This follows quickly, as once we make our assumptions

$$f(x \cdot f(y)) = f(x)f(f(y)) = f(x) \cdot 1/y$$

□

Remark. By Corollary 3 and Theorem 8, the converse of the above also holds.

Let p_i be the i^{th} prime number. Now, every rational number q can be uniquely represented as

$$q = \prod_{i=1}^{\infty} p_i^{k_i}$$

where $k_i \in \mathbb{Z}$ for all i. Furthermore, in this representation only finitely many of the k_i are nonzero, so we do not need to worry about this product being well-defined.

We define $F : \mathbb{Q}^+ \to \mathbb{Q}^+$ to have the quality that $F(1/x) = 1/F(x)$ and $F(xy) = F(x)F(y)$ for all $x, y \in \mathbb{Q}^+$. It follows that $F(x/y) = F(x)/F(y)$, $F(x^n) = (F(x))^n$ for integers n, and $F(1) = 1$. Thus, $F(q)$ can be uniquely determined by the representation of q given above as a product of powers of primes. Specifically,

$$F(q) = \prod_{i=1}^{\infty} \big(F(p_i)\big)^{k_i}$$

Thus we will define F explicitly only on the prime integers, and thus give its full definition on $\mathbb{Q}^+$. Define

$$F(p_i) = \begin{cases} 1/p_{i+1} & i \text{ odd} \\ p_{i-1} & i \text{ even} \end{cases}$$

The motivation is that we now have $F(F(p_i)) = F(1/p_{i+1}) = 1/p_i$ for i odd and $F(F(p_i)) = F(p_{i-1}) = 1/p_i$ for i even.

Now that we have defined F, we need to show that $F(F(q)) = 1/q$. If q has the representation given above

$$\begin{aligned}
F(F(q)) &= F\Big(F\Big(\Big(\prod_{i\ odd} p_i^{k_i}\Big)\Big(\prod_{i\ even} p_i^{k_i}\Big)\Big)\Big) = F\Big(\Big(\prod_{i\ odd}(F(p_i))^{k_i}\Big)\Big(\prod_{i\ even}(F(p_i))^{k_i}\Big)\Big) \\
&= F\Big(\Big(\prod_{i\ odd}(1/p_{i+1})^{k_i}\Big)\Big(\prod_{i\ even}(p_{i-1})^{k_i}\Big)\Big) = \Big(\prod_{i\ odd}(1/F(p_{i+1}))^{k_i}\Big)\Big(\prod_{i\ even}(F(p_{i-1}))^{k_i}\Big) \\
&= \Big(\prod_{i\ odd}(1/p_i)^{k_i}\Big)\Big(\prod_{i\ even}(1/p_i)^{k_i}\Big) = \prod_{i=1}^{\infty}(1/p_i)^{k_i} = 1/q
\end{aligned}$$

where each sum runs over $i \geq 1$.

In conclusion, we have $F : \mathbb{Q}^+ \to \mathbb{Q}^+$ well defined with $F(F(x)) = 1/x$ and $F(xy) = F(x)F(y)$ for all $x, y \in \mathbb{Q}^+$. Thus by Theorem 9 F satisfies $F(x \cdot F(y)) = F(x)/y$ for all $x, y \in \mathbb{Q}^+$.