Perfect Roll and Wheels with Spokes

Clay Thomas

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Abstract

When a ball is rolling down a hill with an arbitrarily large coefficient of friction, it rolls with an acceleration less that it would if it was slipping. Different shapes roll at different accelerations, regardless of the scale of the object. A "wheel with pegs" can produce accelerations arbitrarily close to the slipping-acceleration.

1 General Physics

Throughout the rest of this paper, we will consider an incline elevated to an angle θ above the horizontal. Upon this ramp there will be objects of mass M which roll about an axis with a distance R from the incline.

When a rollable object sits on an incline, there are three forces acting upon it, the weight, the friction, and the normal force. Weight W can be broken into two components, $W_\perp$ perpendicular to the incline and $W_\parallel$ parallel to the incline. $W_\perp$ and N (the normal force) are equal in magnitude and opposite in direction. This leaves two other forces, both parallel to the incline, $W_\parallel$ and the frictional force f. So we have

$$\sum F = Ma = W_\parallel - f = Mg\sin\theta - f$$

and thus

$$a = g\sin\theta - f/M$$

Furthermore, the frictional force produces a torque on the object. All the other forces act on the objects center, and thus have no torque. The frictional force, however, acts at a distance R from the axis of rotation, perpendicular to the radius itself. Thus,

$$\sum \tau = I\alpha = fR.$$

From circular motion we know that $R\alpha = a$, so $\sum \tau = Ia/R$, so we find that

$$a = fR^2/I$$

Now we plunge into the algebra

$$a = g\sin\theta - f/M = fR^2/I$$

$$g\sin\theta = f(R^2/I + 1/M)$$

$$f = \frac{g\sin\theta}{R^2/I + 1/M}$$

and finally

$$a = fR^2/I = \frac{R^2}{I}\frac{g\sin\theta}{R^2/I + 1/M} = \frac{g\sin\theta}{1 + I/(R^2M)}$$

Much of the remainder of the paper is concerned with the quantity

$$\kappa = \frac{I}{R^2M}$$

which determines how close to the optimal slip-acceleration ($g\sin\theta$) the rolling acceleration will be.

2 Some Races

Let us find the moments of inertia of some simple shapes, then plug then into the formula above to find the acceleration those objects will have (depending on the proportions?). In the following calculations, δ is the density and M is the mass, and if applicable, R is the radius and H is the height.

For a solid cylinder,

$$\iiint r^2\delta \mathrm{d}V = \int_0^{2\pi}\int_0^H\int_0^R r^2\delta r \,\mathrm{d}r\mathrm{d}z\mathrm{d}\theta = \frac{1}{2}\pi\delta HR^4 = \frac{1}{2}MR^2$$

For a hollow cylinder,

$$\iint r^2\delta\mathrm{d}\sigma = \pi\delta HR^4 = MR^2$$

And finally, for a solid sphere,

$$\iiint r^2\delta\mathrm{d}V = \int_0^{2\pi}\int_0^{\pi}\int_0^R (\rho\sin\theta)^2\delta\rho^2\sin\phi \,\mathrm{d}\rho\mathrm{d}\phi\mathrm{d}\theta = \frac{8\pi}{15}\delta R^5 = \frac{2}{5}MR^2$$

Thus, for a solid cylinder of mass M and radius R,

$$a = \frac{g\sin\theta}{1 + \frac{MR^2/2}{MR^2}} = \frac{2}{3}g\sin\theta$$

and just like magic, both M and R drop out of the formula. Thus, any solid cylinder, of any size, will have the same acceleration on the same incline.

For a hollow cylinder,

$$\kappa = \frac{I}{R^2 M} = 1$$

$$a = \frac{g \sin\theta}{1+1} = \frac{1}{2} g \sin\theta$$

and for a solid sphere,

$$\kappa = \frac{2}{5}$$

$$a = \frac{g \sin\theta}{1+2/5} = \frac{5}{7} g \sin\theta$$

Thus, the solid sphere, of any size, will win in a race down a hill, because it has the smallest κ.

3 Wheels with Pegs

Given the formula for κ found above, one strategy to increase the acceleration of a rolling object is to decrease the ratio of its moment of inertia to its mass, while holding the radius to the incline constant. One shape that may allow the freedom to do this would be a "wheel with pegs," similar to a flywheel. This wheel would have one central disc and a cylinders on each side, jutting out perpendicular to the disc. The central disc has a radius R and width W, while the cylinders each have radius SR and length LW. Summing the masses and moments of inertia of the three cylinders, you get

$$M = (2\pi(SR)^2(LW) + \pi R^2 W)\delta = \pi\delta(R^2W)(2S^2L+1)$$

$$I = 2\frac{\pi}{2}\delta(SR)^4(LW) + \frac{\pi}{2}\delta(R)^4(W) = \frac{\pi}{2}\delta(R^4W)(2SL+1)$$

Thus

$$\kappa = \frac{\frac{\pi}{2}\delta(R^4W)(2S^4L+1)}{R^2\pi\delta(R^2W)(2S^2L+1)} = \frac{2S^4L+1}{4S^2L+2}$$

Now, we will investigate strategies for minimizing theta, and thus maximizing acceleration.

$$\lim_{L\to\infty} \kappa = \lim_{L\to\infty} \frac{2S^4 + L^{-1}}{4S^2 + 2L^{-1}} = \frac{S^2}{2}$$

Thus, increasing the length of the pegs will always decrease κ, but not beyond a specific limit defined by the radius of the cylinders.

Now we know how to "minimize" κ by manipulating length, but what if we have a fixed length proportion L? In this case, there is a specific radius proportion S that produces the highest acceleration. To do this, we derive κ and manipulate and plug back in the result.

$$\frac{\partial\kappa}{\partial S} = \frac{(8S^3L)(4S^2L+2)-(2S^4L+1)(8SL)}{(4S^2L+2)^2}$$

which passes from negative to positive when

$$2S^4L+2S^2-1$$

passes from negative to positive. This is when

$$S^2 = S^2_{min} = \frac{\sqrt{1+2L}-1}{2L}$$

at which point

$$\kappa = \kappa_{min} = \frac{1+2L-\sqrt{1+2L}}{2L\sqrt{1+2L}}$$

Finally, note that

$$\lim_{L\to\infty}\kappa_{min} = \lim_{L\to\infty}\frac{L^{-3/2}+2L^{-1/2}-\sqrt{L^{-3}+2L^{-2}}}{2\sqrt{L^{-1}+2}} = \frac{0}{2\sqrt{2}} = 0$$

Thus we have a shape that we can manipulate to have acceleration arbitrarily close to that of a slipping object by increasing the L and setting S at the corresponding value.